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Radiative Shape Factors Between Differential Ring Elements on Concentric Axisymmetric Bodies

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Nomenclature

dA_i	= area of differential ring element
B	= function defined by Eq. (9)
dF_{d1-d2}	= infinitesimal radiation shape factor from differential element dA_1 to dA_2
ds_i	= width of differential ring element $dA_i = 2\pi r_i ds_i$
r, r_1, r_2	= radius (of element 1 and 2, respectively)
$R_i(z), R_o(z)$	= local radius of inner and outer axisymmetric body
S	= distance between two points on dA_1 and dA_2
z, z_1, z_2	= axial position of element 1 and 2, respectively
α_{12}	= function defined by Eq. (6)
β_i	= angle between surface normal to dA_i and point-connection line S
$\theta, \theta_1, \theta_2$	= tilt angle of surface with respect to z-axis
$\psi, \psi_{\min}, \psi_{\max}$	= (minimum or maximum) azimuth angle with which strip dA_2 is seen from a point on dA_1
ϕ_1, ϕ_2	= function defined by Eq. (6)
Γ_i, Γ_o	= minimum and maximum permissible values for $\cos\psi$ based on interfering surface between any two differential ring elements

Introduction

LARGE amounts of radiative shape factors have been published in the past, in particular during the 1960's, many in the form of formulas, some in the form of computer calculations and graphs. A good review on published shape factors has been given by Siegel and Howell[1] and Howell[2]. An analytical formula has been given by Morizumi[3] for a simple paraboloidal surface, while some numerical calculations have been carried out by Robbins and Todd[4] for a single axisymmetric body. Chung and Naraghi[4,5] formulated shape factor expressions between a sphere and a number of axisymmetric bodies, while Masuda[7] treated the case of circular-finned cylinders. It is the purpose of this paper to add working formulas for shape factors between two ring strip elements on two arbitrarily shaped concentric axisymmetric bodies.

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Analysis

Figure 1 shows a schematic of the plasma chamber of the NET (next European torus) fusion reactor as an example for two concentric axisymmetric bodies. Consider two infinitesimal bands ds_1 and ds_2, as indicated in Fig. 1 for the case that ds_1 lies on the outer axisymmetric body, while ds_2 lies on the inner one. The shape factor between them may be evaluated as[1]

$$dF_{d1\to d2} = \frac{2}{\pi}\int_{\psi_{\min}}^{\psi_{\max}} \frac{\cos\beta_1 \cos\beta_2}{S^2}\, d\psi\, r_2\, ds_2 \tag{1}$$

where symmetry with respect to the azimuthal angle ψ has been incorporated. Here, S is the distance between two points on the bands, β_i the angle between the surface normal of ds_i and the vector to the point on the other band, ψ the azimuthal angle between the two points (in the plane perpendicular to the rotation axis), and $\psi_{\min}$ and $\psi_{\max}$ the limiting angles with which the band ds_2 is seen from a point on ds_1. To clarify the meaning of the limiting angles $\psi_{\min}$ and $\psi_{\max}$, Fig. 2 shows vertical and horizontal cuts through the concentric axisymmetric bodies depicted in Fig. 1. The locations of dA_1 and dA_2 have been changed a bit in order to show certain shading effects. For $\psi = 0$, a vector from dA_1 to dA_2 would intersect A_1 itself before getting to dA_2. Thus, there is a minimum azimuthal angle $\psi_{\min}$ at which the vector will just graze by the corner at A_1. If there were no inner cylinder $\psi_{\max}$ would be determined by the range of ψ over which $\cos\beta_1$ and $\cos\beta_2$ would remain positive (e.g., $\psi_{\max} = \pi$ for a horizontal ring and $\psi_{\max} = \cos^{-1}(r_2/r_1)$ for a vertical ring). With an inner cylinder present, a vector from dA_1 to dA_2 with an azimuthal angle larger than the one labeled $\cos^{-1}\Gamma_i$ would intersect the inner cylinder before getting to dA_2. The relevant $\psi_{\max}$ would then be the smaller of the two, as indicated in Fig. 2. The integrand in Eq. (1) is readily found from geometric considerations as

$$S^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\psi + (z_2 - z_1)^2 \tag{2}$$

$$S\cos\beta_1 = -(r_1 - r_2\cos\psi)\cos\theta_1 - (z_2 - z_1)\sin\theta_1 \tag{3}$$

$$S\cos\beta_2 = (r_1\cos\psi - r_2)\cos\theta_2 + (z_2 - z_1)\sin\theta_2 \tag{4}$$

where θ_i is the angle between the z axis and strip ds_i as indicated in Fig. 2 (measured from the z axis into the outward direction onto the backside of the surface; thus, $\theta = 0$ for a vertical, outward facing strip, $-\pi/2 < \theta < \pi/2$ for outward facing strips, and $\pi/2 < \theta < 3\pi/2$ for inward facing strips). Therefore,

$$\frac{dF_{d1\to d2}}{2\pi r_2 ds_2} = \frac{\cos\theta_1\cos\theta_2}{4\pi^2 r_1 r_2}\int_{\psi_{\min}}^{\psi_{\max}} \frac{(\phi_1 - \cos\psi)(\phi_2 - \cos\psi)}{(\alpha_{12} - \cos\psi)^2}\, d\psi \tag{5}$$

with

$$\phi_i = \frac{r_i}{r_j} + \frac{z_j - z_i}{r_j}\tan\theta_i,\quad i=1, j=2, \text{ or } i=2, j=1$$

$$\alpha_{12} = \tfrac{1}{2}\left(\frac{r_1}{r_2} + \frac{r_2}{r_1}\right) + \frac{(z_2 - z_1)^2}{2r_1 r_2} \tag{6}$$

This may be integrated to yield

$$\frac{dF_{d1\to d2}}{2\pi r_2 ds_2} = \frac{\cos\theta_1\cos\theta_2}{4\pi^2 r_1 r_2}[B(\alpha_{12},\phi_1,\phi_2,\cos\psi_{\max}) - B(\alpha_{12},\phi_1,\phi_2,\cos\psi_{\min})] \tag{7}$$

Table 1 Limiting values for $\cos\psi_{min}$ and $\cos\psi_{max}$ in Eq. (1) (if $\cos\psi_{min} < \cos\psi_{max}$ then $dF_{d1-d2} \equiv 0$)

	$\cos\theta_1 \geq 0$	$\cos\theta_1 \leq 0$
$\cos\theta \geq 0$	$\cos\psi_{min} = \min(\Gamma_o,1)$ $\cos\psi_{max} = \max(\phi_1,\phi_2,\Gamma_i,-1)$	$\cos\psi_{min} = \min(\phi_1,\Gamma_o,1)$ $\cos\psi_{max} = \max(\phi_2,\Gamma_i,-1)$
$\cos\theta_2 \leq 0$	$\cos\psi_{min} = \min(\phi_2,\Gamma_o,1)$ $\cos\psi_{max} = \max(\phi_1,\Gamma_i,-1)$	$\cos\psi_{min} = \min(\phi_1,\phi_2,\Gamma_o,1)$ $\cos\psi_{max} = \max(\Gamma_i,-1)$

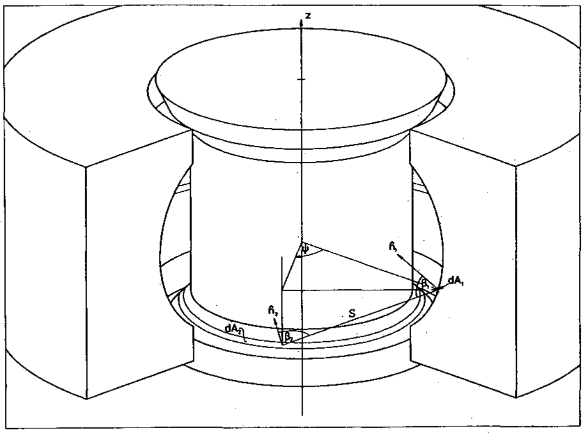

Fig. 1 Schematic of two concentric axisymmetric bodies for shape factor evaluation.

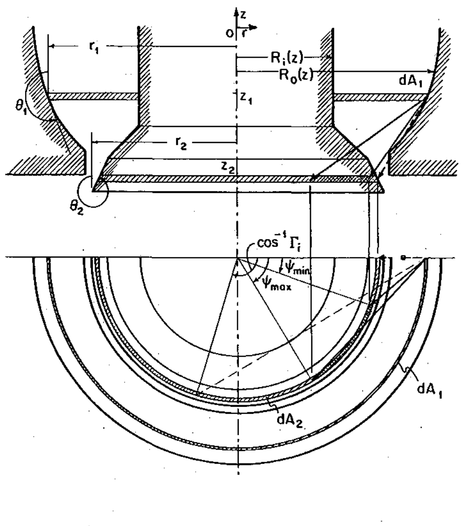

Fig. 2 Two-dimensional view of axisymmetric bodies (cuts along and perpendicular to axis).

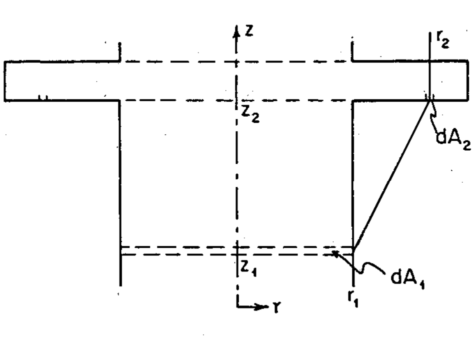

Fig. 3 Geometry of circular-finned cylinder example.

with

$$B(\alpha,\phi_1,\phi_2,\phi_e) = \cos^{-1}\phi_e + \frac{(\alpha-\phi_1)(\alpha-\phi_2)}{\alpha^2-1}\frac{(1-\phi_e^2)^{\frac{1}{2}}}{\alpha-\phi_e} + 2\frac{2\alpha-\phi_1-\phi_2-\alpha(\alpha^2-\phi_1\phi_2)}{(\alpha^2-1)^{3/2}} \times \tan^{-1}\left[\left(\frac{\alpha+1}{\alpha-1}\frac{1-\phi_e}{1+\phi_e}\right)^{\frac{1}{2}}\right], \quad \alpha \neq 1$$
$$= \cos^{-1}\phi_e, \quad \alpha = 1 \tag{8}$$

It remains to determine the limiting angles ψ_{min} and ψ_{max}. In the absence of obstructions between the two rings (i.e., other parts of the axisymmetric bodies), these limiting angles follow from the conditions that

$$\cos\beta_i \geq 0, \; i = 1,2 \tag{9}$$

Depending on the signs of the $\cos\theta_i$, as well as on the magnitudes of ϕ_1 and ϕ_2, either ϕ_1 or ϕ_2 or both may limit the range of allowable $\cos\psi$. For example, if both, $\cos\theta_1 > 0$ and $\cos\theta_2 > 0$ (both strips face to the outside), strip 1 cannot see the top of strip 2 if either $\phi_1 > 1$ or $\phi_2 > 1$, strip 1 can see all of strip 2 if $\phi_1 < -1$ and $\phi_2 < -1$, and strip 1 can see strip 2 over a ψ range defined by $\max(\phi_1,\phi_2) < \cos\psi < 1$ for other values of ϕ_1 and ϕ_2. The $\cos\psi$ ranges for all possible situations are summarized in Table 1 (including the obstruction effects discussed below). For the two strips shown in Fig. 2, one finds $\pi < \theta_1 < 3/2\pi$ and $3/2\pi < \theta < 2\pi$. Thus, from Table 1 with $\cos\theta_1 < 0$ and $\cos\theta_2 > 0$ and barring obstructions between the two strips, $\cos\psi_{min}$ is the maximum of ϕ_1 and 1, while $\cos\psi_{max}$ is the maximum of ϕ_2 and -1. It is easily seen from Fig. 2 (by extending the surface tangent at dA_1) that $\phi_1 > 1$, indicating that $\cos\beta_1 > 0$ for $\psi = 0$ and, therefore, $\psi_{min} = 0$ if no obstructions lie between the strips. Similarly, one readily can see that $0 < \phi_2 < 1$ so that $\cos\psi_{max} = \phi_2$ (again barring obstruction between the rings).

Since both strips are on the surfaces of two concentric axisymmetric bodies, the view from a point on ds_1 to a part of ds_2 is probably obstructed by the inner axisymmetric body and may also be partially obstructed by the outer body. Mathematically, one may state that the shortest distance between a vector from ds_1 to ds_2 and the rotation axis may nowhere be smaller than the local radius of the inner body $R_i(z)$ and may nowhere be larger than the local radius of the outer body $R_o(z)$. This may be expressed as

$$\cos\psi \geq \Gamma_i = \max\left[\frac{R_i^2(z)(z_2-z_1)^2 - r_1^2(z_2-z)^2 - r_2^2(z-z_1)^2}{2r_1r_2(z-z_1)(z_2-z)}\right]_{z\in(z_1,z_2)} \tag{10}$$

and

$$\cos\psi \leq \Gamma_o = \min\left[\frac{R_o^2(z)\,(z_2-z_1)^2 - r_1^2(z_2-z)^2 - r_2^2(z-z_1)^2}{2r_1r_2(z-z_1)\,(z_2-z)}\right]_{z\in(z_1,z_2)} \tag{11}$$

In general, the maximum and minimum over the interval (z_1,z_2), respectively, has to be found in order to determine Γ_i and Γ_o. Often, it is obvious from the geometry at what location z the function in Eq. (10) has its maximum or where the function in Eq. (11) has its minimum. For example, for the case depicted in Fig. 2, the minimum of the function in Eq. (11) (and, thus, $\psi_{\min}$) is obviously determined by the lower corner on the outer body where the circular cross section ends and the vertical piece begins, as indicated in the figure. If the inner body is a cylinder (or at least the obstructing part of the inner body is cylindrical), the maximization is readily carried out by looking at the projection of the vector from $\mathrm{d}s_1$ to $\mathrm{d}s_2$ onto a cross section of the cylinder, leading to

$$\Gamma_i = \frac{R_i^2}{r_1 r_2} - \left[\left(1-\frac{R_i^2}{r_1^2}\right)\left(1-\frac{R_i^2}{r_2^2}\right)\right]^{\frac{1}{2}} \tag{12}$$

Even for somewhat more complicated geometries such as the one in Fig. 2, this relationship may be employed: no vector from $\mathrm{d}A_1$ to $\mathrm{d}A_2$ with a positive $\cos\beta_2$ (i.e., hitting $\mathrm{d}A_2$ from the top) could intersect either one of the conical pieces, but it could intersect the cylinder, making Eq. (12) valid. This is indicated in Fig. 2 as $\psi = \cos^{-1}\Gamma_i$; in the present case, this angle is larger than the $\psi_{\max}$ determined from the $\cos\beta_2 > 0$ condition, and therefore, does not apply.

In summary, the radiation shape factor between two strips located on the same or opposite surfaces of two concentric axisymmetric bodies is determined by Eqs. (7) and (8) with the limiting values $\cos\psi_{\max}$ and $\cos\psi_{\min}$ taking on the values ϕ_1, ϕ_2, Γ_i, and Γ_o depending on the location and orientation of the strips. A detailed listing of the values for $\cos\psi_{\max}$ and $\cos\psi_{\min}$ for all situations is given in Table 1. Only those combinations of values for ϕ_1 and ϕ_2 are included in the table that result in nonzero shape factors (for example, if $\cos\theta_1 > 0$ and $\cos\theta_2 > 0$, the strips cannot see each other at all if $\phi_1 > 1$ and $\phi_2 > 1$).

As an additional numerical example, consider the case of an infinitesimal strip on a cylinder and a second strip on a disk attached perpendicularly to the cylinder (Fig. 3). This corresponds to the special case of circular-finned cylinders for which Masuda[7] has already given analytical expressions. For a cylinder as the inner axisymmetric body, we have $\theta_1 = 0$ and, for a horizontal fin (above $\mathrm{d}s_1$, pointing down), we have $\theta_2 = \pi/2$. With $\Delta z = z_2 - z_1 > 0$, we have $\phi_1 = r_1/r_2$ and $\phi_2 \to \pm\infty$ (for $\theta_2 = \pi/2 \pm 0$). There are no obstructions between the two strips, so Γ_i and Γ_o do not apply. It follows from Table 1 that $\cos\psi_{\min} = 1$ and $\cos\psi_{\max} = \phi_1$ (note that both $\cos\theta_2 \geq 0$ with $\phi_2 \to -\infty$ and $\cos\theta_2 \leq 0$ with $\phi_2 \to +\infty$ lead to the same result). For $\psi_{\min} = 0$ it follows that $B(\alpha,\phi_1,\phi_2,\cos\psi_{\min}=1) = 0$ and

$$\lim_{\theta_2\to 0} \cos\theta_2 B(\alpha,\phi_1,\phi_2,\cos\psi_{\max}=\phi_1) = \frac{\Delta z}{r_1}\left\{\frac{(1-\phi_1^2)^{\frac{1}{2}}}{\alpha^2-1} + 2\frac{1-\alpha\phi_1}{(\alpha^2-1)^{3/2}}\tan^{-1}\left[\left(\frac{\alpha+1}{\alpha-1}\frac{1-\phi_1}{1+\phi_1}\right)^{\frac{1}{2}}\right]\right\}$$

Thus, the shape factor is

$$\mathrm{d}F_{\mathrm{d}1-\mathrm{d}2} = \frac{\Delta z\,\mathrm{d}s_2}{2\pi r_1^2}\left\{\frac{(1-\phi_1^2)^{\frac{1}{2}}}{\alpha^2-1} + 2\frac{1-\alpha\phi_1}{(\alpha^2-1)^{3/2}}\tan^{-1}\left[\left(\frac{\alpha+1}{\alpha-1}\frac{1-\phi_1}{1+\phi_1}\right)^{\frac{1}{2}}\right]\right\} \tag{13}$$

It is readily verified that this formula is identical to Eq. (13) in the paper by Masuda.

Conclusion

A simple analytical formula has been given for the radiative shape factor between any two ring strips placed on two arbitrarily shaped concentric axisymmetric bodies. This approach eliminates the need for one numerical quadrature with obstruction checking for radiative heat-transfer calculations in such geometries.

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Perturbation Solution for Spherical Solidification by Convective Cooling

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Nomenclature

c	= specific heat of solid material
h	= heat transfer coefficient
k	= thermal conductivity of solid material
L	= latent heat of fusion
R	= radial position in the solidified material
R_f	= radial position of the freezing front
R_0	= radius of sphere
T	= temperature in the solidified material
T_f	= freezing temperature
T_∞	= temperature of cooling fluid
t	= time
α	= thermal diffusivity of solid material
ρ	= density of solid material

Introduction

THE problem of the inward solidification of spheres has received considerable attention in the literature.[1-5] The

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